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Technical Comments

Comment on "Simple Equations for Helical Vortex Wakes"

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WOOD[1] has presented a formulation for the efficiency of a propeller or rotor that has an infinite number of blades or, equivalently, is represented by an actuator disc, based, respectively, on simple vortex and momentum theory. In this work, an arbitrary "thermodynamic efficiency" is introduced to account for irreversible conversions of energy by viscosity which are said to be omitted in "most 'aerodynamic' analyses."

Long-established methods of propeller analysis[2-4] give for η, the propeller efficiency of a blade element

$$\eta = \frac{(1 - a')}{(1 + a)} \frac{\tan \phi}{\tan(\phi + \gamma)}$$

where a is the axial induced velocity factor in the far wake, a' is the radial induced velocity factor in the far wake such that the induced axial velocity is aV and the induced tangential velocity is $a'\Omega r$. Also,

$$\tan \phi = \frac{V}{\Omega r} \frac{1 + a}{1 - a'}$$

$$\tan \gamma = \frac{C_d}{C_l}$$

where V is the forward velocity, Ω is the angular rotational speed, r is the radius distance from the axis of rotation, C_d is the propeller section profile drag coefficient, and C_l is the propeller section lift coefficient. Thus, it is clear that an efficiency η_s, arising from the action of the propeller blades, represents the major contribution to the undefined irreversible energy losses of Ref. 1, where

$$\eta_s = \frac{\tan \phi}{\tan(\phi + \gamma)}$$

Similar results can be obtained for hovering rotors. Glauert[3] showed that for a representative propeller, the calculated values of integrated efficiency associated with profile drag losses were between 80–90% over most of the operating range. Using calculated values in an example given by Weick,[4] values of η_s between 0.85–0.92 are obtained for blade sections from 0.3 to 0.90 of the radius. It is to be noted that the values of thermodynamic efficiency obtained empirically from test results in Fig. 2 of Ref. 1 lie in these ranges.

The need to account for vortex-induced flow velocity in the wake of propellers and rotors in determining the helical pitch of the vortices as indicated in Fig. 1 of Ref. 1 is also generally recognized. Theodorsen[5] pointed out that this is an essential element in the application of the theory of optimum propeller load distribution to heavily loaded propellers. Also, it has been shown[6] that the vortex and momentum theories for hovering rotors and propellers having infinite numbers of blades lead to identical results only when the effect of vortex-induced velocity on the helical pitch in the far wake is accounted for in the vortex theory.

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Reply by the Author to A. H. Flax

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CONVENTIONAL analyses of propellers or rotors, leading, e.g., to the equations given by Flax,[1] are based on the one-dimensional conservation equations for mass, momentum, angular momentum, and energy for the airflow through the blade disc. The main difficulty appears to be the exclusion of the possibility that significant amounts of the conserved quantities reside in the trailing vortices, in particular the tip vortices.

The purpose of the Note was to consider Eq. (1), relating p_∞, the pitch of the tip vortices *behind a finite number of blades*, to the average axial velocity in the far-wake U_∞, within the framework of conventional analyses. To the author's knowledge, Eq. (1) is new, although it is an immediate consequence of Eq. (8) of Hardin.[2] The primary outcome of the Note is Eq. (6), a relation between p_∞ and the advance ratio J, derived from Eq. (1) using the assumption that some of the energy and angular momentum supplied by the blades

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resides within the vortices. As shown by the dotted line in Fig. 1, this simple relation agrees remarkably well with data from propeller wakes. If the assumption is removed for the angular momentum, then the agreement deteriorates significantly, as shown by the solid circles in Fig. 1. If the assumption was unnecessary for the energy, then the thermodynamic efficiency η in Fig. 2 would have been unity.

If only the energy equation is considered, a reasonable alternative to the assumption of the last paragraph is to expect that η accounts for the action of viscosity in dissipating mechanical energy into internal energy within the near wake, a process that is not otherwise included in the energy equation. There would then be a relationship between the efficiencies described by Flax, which involve the drag of the blade, and η. However, the need for η in the angular momentum equation cannot be explained in this manner because the viscous destruction of angular momentum generally takes much longer, witness the persistence of trailing vortices and even the notion of a far wake.

There are several other related possibilities for the necessity of including η:

1) It is a factor that partially compensates for errors in assuming that the trailing vorticity comprises only tip and hub vortices, and, therefore, U_∞ is uniform. It was suggested in the Note that the reason for the poorer agreement between Eq. (8) and data for hovering rotors is that their wakes are much less uniform than propeller wakes.

2) The use of average velocities in the nonlinear conservation equations for momentum, angular momentum, and energy. The resulting equations, therefore, ignore terms that arise in a similar manner to the Reynolds stresses in the averaged equations for turbulent flows. In an attempt to avoid this problem, Wood[3] modeled the tip vortices of a wind turbine at high tip speed ratio (low J) as a series of vortex rings and applied the conservation equations in a frame of reference moving with the convection velocity of the rings. This velocity, which depends partly on the structure of tip vortex, appeared in the final momentum and energy equations. The extreme case of runaway, where a turbine extracts no energy, but the thrust and wake expansion are maximized, apparently occurs when the tip vortices contain nearly all the energy extracted from the wake.[4]

3) The tip vortices in some circumstances contain significant amounts of momentum, angular momentum, and energy. In other words, the wake is not force-free.

Of the three possibilities, the last is the most fascinating.

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